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On groups with finitely many Conradian orderings

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Abstract

We study the space of left-orderings on groups with (only) finitely many Conradian orderings. We show that, within this class of groups, having an isolated left-ordering is equivalent to having finitely many left-orderings.

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Introduction

A (total) left-ordering $\preceq$ on a group G is said to be *isolated* if there is a finite family $\{g_1, \ldots, g_n\} \subset G$ such that $\preceq$ is the only left-ordering on G with the property that $g_i \succ id$, for $1 \leq i \leq n$. This criteria may be used to define a topology on $\mathcal{LO}(G)$, the set of all left-orderings on G. It was proved by Sikora in [15] that with this topology, $\mathcal{LO}(G)$ is a totally disconnected, Hausdorff and compact topological space. Moreover, when G is countable, this topology is metrizable. See §1.1 for further details.

Knowing whether a given group has an isolated left-ordering has been a question of major interest in the recent development of the theory of orderable groups. A big progress was made by Tararin who classified left-orderable groups that admit only finitely many left-orderings (a *Tararin group*, for short), see Theorem 1.3 or [8, §5.2].

Albeit Tararin's description has shown to be very useful, the comprehension of groups admitting isolated left-orderings is far from being reached. Some progress in this direction was done in [4] and [10]. In [4], Dubrovina and Dubrovin show that braid groups have isolated left-orderings, whereas in [10], Navas describes a family of two-generated groups (which contains the three strands braid group B_3) having infinitely many left-orderings together with isolated left-orderings. For a nice survey about orderings on braid groups, see [3].

It follows from Tararin's description that every Tararin group is solvable. On the other hand, neither braid groups nor the groups described in [10] are solvable. Moreover, in [12] it is shown that the only nilpotent groups having isolated left-orderings are the torsion-free, rank-one Abelian groups[1]. Thus, it is natural to pose the

Main Question: Is it true that, in the class of left-orderable solvable groups, having an isolated left-ordering is equivalent to having only finitely many left-orderings?

[1] Recall that a torsion-free Abelian group Γ has *rank* n if n is the least integer for which Γ embeds into $\mathbb{Q}^n$.

In this work we give a partial (affirmative) answer to this question. Recall that a left-ordering $\preceq$ on a group G is *Conradian* (or a C-ordering) if $f \succ id$ and $g \succ id$ imply $fg^2 \succ g$, see [2, 7, 12].

Main Theorem: *Let G be a group admitting only finitely many C-orderings. Then G either admits only finitely many left-orderings (so G is a Tararin group) or has no isolated left-orderings.*

We note that the relation between left-orderings and Conradian orderings is much deeper than just the one described in the Main Theorem. For instance, in [12, §4] it is proved that no Conradian ordering is isolated in a group with infinitely many left-orderings, and also a criterion is given for a left-ordering to be isolated in terms of the so-called *Conradian soul* of an ordering. Nevertheless, we will not make use of those facts in this work.

To prove the Main Theorem we will make use of the algebraic description of groups admitting (only) finitely many C-orderings, here Theorem 1.2, which was obtained in [14]. As shown in Theorem 1.2, groups with finitely many Conradian orderings admits a unique rational series (see definition below), and our proof proceeds by induction on the length of this series. In §2, we explore the (initial) case of groups with rational series of length two. In this case, we give an explicit description of $\mathcal{LO}(G)$. In §3.1 we obtain some technical results concerning the action of inner automorphisms of a group G with a finite number of Conradian orderings. As a consequence, we show that the maximal convex subgroup of G (with respect to a C-ordering) is a group that fits into the classification made by Tararin. Finally, in §3.2, we prove the general case, while §3.3 is devoted to the description of an illustrative example.

1 Preliminaries

We begin this section recalling the foundational result [2, Theorem 4.1]. Recall that in a left-ordered group G, G_g (resp. G^g) denotes the maximal (resp. minimal) convex subgroup which does not contain (resp. contains) $g \in G$. (A subset S of a left-ordered group Γ is said to be convex if and only if for every $\gamma \in \Gamma$ such that $s_1 \preceq \gamma \preceq s_2$, for some $s_1,\ s_2$ in S, we have that $\gamma \in S$.)

Theorem 1.1 (Conrad)**.** *An ordering $\preceq$ on a group G is Conradian if and only if for every $g \in G$, $g \neq id$, we have that G_g is normal in G^g, and there exists a unique up to multiplication by a positive real number, non-decreasing group homomorphism $\tau_{\preceq}^{g} \colon G^g \to \mathbb{R}$ whose kernel coincides with G_g.*

The Conrad Theorem implies that any C-orderable group is locally indicable[2], and a remarkable result from [1] shows that the class of C-orderable groups coincides with the class of locally indicable groups, see also [12]. Thus, all torsion-free, one-relator groups are C-orderable [1, 6].

[2]A group G is locally indicable if for any finitely generated subgroup H there is a non-trivial group homomorphism from H to the real numbers under addition.

In [14] a structure theorem was given for groups admitting only finitely many Conradian orderings. For the statement, recall that a series

$$\{id\} = G_0 \lhd G_1 \lhd \ldots \lhd G_{n-1} \lhd G_n = G$$

is said to be *rational* if it is subnormal (*i.e.,* each G_i is normal in G_{i+1}) and each quotient G_{i+1}/G_i is torsion-free rank-one Abelian. We say that the rational series is *normal* if, in addition, $G_i \lhd G$ for all $1 \leq i \leq n$.

Theorem 1.2. *Let G be a C-orderable group. If G admits only finitely many C-orderings, then G admits a unique (hence normal) rational series. In this series, no quotient G_{i+2}/G_i is Abelian. Conversely, if G is a group admitting a normal rational series*

$$\{id\} = G_0 \lhd G_1 \lhd \ldots \lhd G_{n-1} \lhd G_n = G$$

so that no quotient G_{i+2}/G_i is Abelian, then the number of C-orderings on G equals 2^n.

One of the crucial steps in proving Theorem 1.2 consist in using the Conrad Theorem to show that in any C-ordering of G -a group with only finitely many Conradian orderings- and any $g \in G$, we have that $G_g = G_i$ and $G^g = G_{i+1}$ for some $0 \leq i \leq n-1$. In particular, in a group with only finitely many Conradian orderings, the convex series given by a C-ordering coincides with the rational series of G.

A sub-class of the class of groups admitting only finitely many Conradian orderings is the class of groups admitting only finitely many left-orderings. This latter class was described by Tararin, [8, §5.2]. Since we will make use of this description, we quote Tararin's theorem below. For the statement, recall that a left-ordering $\preceq$ on a group G is said to be *bi-invariant* (or *bi-ordering*, for short) if $g \succ id$ implies $hgh^{-1} \succ id$ for all $h \in G$. Clearly, every bi-ordering is Conradian.

Theorem 1.3 (Tararin)**.** *Let G be a left-orderable group. If G admits only finitely many left-orderings, then G admits a unique (hence normal) rational series. In this series, no quotient G_{i+2}/G_i is bi-orderable. Conversely, if G is a group admitting a normal rational series*

$$\{id\} = G_0 \lhd G_1 \lhd \ldots \lhd G_{n-1} \lhd G_n = G$$

so that no quotient G_{i+2}/G_i is bi-orderable, then the number of left-orderings on G equals 2^n.

Note that the statement of Tararin's theorem is the same as the statement of Theorem 1.2 though changing 'C-orderings' by 'left-orderings', and the condition 'G_{i+2}/G_i non-Abelian' by 'G_{i+2}/G_i non-bi-orderable'.

1.1 The space of left-orderings of a group

Recall that given a left-ordering $\preceq$ on a group G, we say that $f \in G$ is $\preceq$*-positive* or simply *positive* (resp. $\preceq$*-negative* or *negative*) if $f \succ id$ (resp. $f \prec id$). We denote $P_\preceq$ the set of $\preceq$-positive elements in G. Clearly, $P_\preceq$ satisfies the following properties:

(*i*) $P_\preceq P_\preceq \subseteq P_\preceq$, that is, $P_\preceq$ is a semigroup;

(ii) $G = P_{\preceq} \sqcup P_{\preceq}^{-1} \sqcup \{id\}$, where the union is disjoint, and $P_{\preceq}^{-1} = \{g^{-1} \in G \mid g \in P_{\preceq}\} = \{g \in G \mid g \prec id\}$.

Moreover, given any subset $P \subseteq G$ satisfying the conditions (i) and (ii) above, we can define a left-ordering $\preceq_P$ by $f \prec_P g$ if and only if $f^{-1}g \in P$. Therefore, describing a left-ordering is equivalent to describing its set of positive elements. We usually identify $\preceq$ with $P_{\preceq}$.

Given a left-orderable group G (of arbitrary cardinality), we denote the set of all left-orderings on G by $\mathcal{LO}(G)$. This set has a natural topology first introduced by Sikora for the case of countable groups [15]. This topology can be defined by identifying $P \in \mathcal{LO}(G)$ with its characteristic function $\chi_P \in \{0,1\}^G$. In this way, we can view $\mathcal{LO}(G)$ embedded in $\{0,1\}^G$. This latter space, with the product topology, is a Hausdorff, totally disconnected, and compact space. It is not hard to see that (the image of) $\mathcal{LO}(G)$ is closed inside, and hence compact as well (see [12, 15] for details).

A basis of neighborhoods of $\preceq$ in $\mathcal{LO}(G)$ is the family of the sets $V_{f_1,\ldots,f_k}$ of all left-orderings $\preceq'$ on G such that all the f_i are $\preceq'$-positive, where $\{f_1, \ldots, f_k\}$ runs over all finite subsets of $\preceq$-positive elements of G. Hence, a left-ordering of G is *isolated* (in the sense of the introduction) if an only if it is an isolated point of $\mathcal{LO}(G)$. The (perhaps empty) subspaces $\mathcal{BO}(G)$ and $\mathcal{CO}(G)$ of bi-orderings and C-orderings on G respectively, are closed inside $\mathcal{LO}(G)$, hence compact; see [12].

If G is countable, then this topology is metrizable: given an exhaustion $G_0 \subset G_1 \subset \ldots$ of G by finite sets, for different $\preceq$ and $\preceq'$, we may define $dist(\preceq, \preceq') = 1/2^n$, where n is the first integer such that $\preceq$ and $\preceq'$ do not coincide on G_n. If G is finitely generated, we may take G_n as the ball of radius n with respect to a fixed finite system of generators.

1.2 A basic construction for producing new left-orderings

In this section we describe some basic constructions for creating new left-orderings starting with a given one. The main idea is to exploit the flexibility given by the convex subgroups.

Let $\preceq$ be a left-ordering on a group G. If C is a proper convex subgroup of G, then $\preceq$ induces a total order $\preceq^C$ on the set of left-cosets of C by

$$g_1 C \prec^C g_2 C \Leftrightarrow g_1 c_1 \prec g_2 c_2 \text{ for all } c_1, c_2 \text{ in C.} \tag{1}$$

More importantly, this order is preserved by the left action of G; see for instance [8, §2]. In particular, if C is a normal subgroup, then $\preceq^C$ becomes a left-ordering of the group G/C.

As the reader can easily check, the left-ordering $\preceq$ can be recovered from the left-ordering $\preceq^C$ and the left-ordering $\preceq_C$, defined as the restriction of $\preceq$ to C, by the following equation:

$$g \succ id \Leftrightarrow \begin{cases} gC \succ^C C \text{ or} \\ gC = C \quad \text{and } g \succ_C id. \end{cases}$$

This easily implies

Lemma 1.4. *Let $\preceq$ be a left-ordering on a group G, and suppose there is a non-trivial convex subgroup C. Then there is a continuous injection*

$$\varphi : \mathcal{LO}(C) \to \mathcal{LO}(G)$$

such that $\preceq$ belongs to the image of φ.

Moreover, if in addition C is normal, then we have a continuous injection

$$\varphi : \mathcal{LO}(G/C) \times \mathcal{LO}(C) \to \mathcal{LO}(G)$$

such that $\preceq$ belongs to the image of φ.

Corollary 1.5. *If for a left-ordering $\preceq$ on a group G there is a convex subgroup C such that either C has no isolated left-orderings or such that C is normal and G/C has no isolated left-orderings, then $\preceq$ is non-isolated.*

2 On groups with a rational series of length two

Throughout this section, G will denote a left-orderable, non-Abelian group with a rational series of length 2:

$$\{id\} = G_0 \lhd G_1 \lhd G_2 = G.$$

If the group G is not bi-orderable, then G has a normal rational series of length 2 and the quotient $G_2/G_0 = G$ is non-bi-orderable. Thus G fits into the classification made by Tararin, so it has only finitely many left-orderings.

For the rest of this section we will assume that G is not a Tararin group, so G is bi-orderable. We have

Lemma 2.1. *The group G satisfies that $G/G_1 \simeq \mathbb{Z}$.*

Proof: Consider the action by conjugation $\alpha : G/G_1 \to Aut(G_1)$ given by $\alpha(gG_1)(h) = ghg^{-1}$. Since G is non-Abelian, we have that this action is non-trivial, *i.e.* $Ker(\alpha) \neq G/G_1$. Moreover, $Ker(\alpha) = \{id\}$, since in the other case, as G/G_1 is rank-one Abelian, we would have that $(G/G_1)/Ker(\alpha)$ is a torsion group. But the only non-trivial, finite order automorphism of G_1 is the inversion, which implies that G is non-bi-orderable, thus a Tararin group.

The following claim is elementary and we leave its proof to the reader.

Claim: If Γ is a torsion-free, rank-one Abelian group such that $\Gamma \not\simeq \mathbb{Z}$, then for any $g \in \Gamma$, there is an integer $n > 1$ and $g_n \in \Gamma$ such that $g_n^n = g$.

Now take any $b \in G \setminus G_1$ so that $\alpha(bG_1)$ is a non-trivial automorphism of G_1. Since G_1 is rank-one Abelian, for some positive $r = p/q \in \mathbb{Q}$, $r \neq 1$, we must have that $bab^{-1} = a^r$ for all $a \in G_1$. Suppose that $G/G_1 \not\simeq \mathbb{Z}$. By the previous claim, we have a sequence of increasing integers $(n_1, n_2 \ldots)$ and a sequence $(g_1, g_2, \ldots)$ of elements in G/G_1 such that $g_i^{n_i} = bG_1$. In particular we have that $g_i a g_i^{-1} = a^{r_i}$, where r_i is a rational such that $r_i^{n_i} = r$. In other words, given r, we have found among the rational numbers, an infinite collection of r_i solving the equation $x^{n_i} - r = 0$, but, by the Rational Roots Theorem or Rational Roots Test [9, Proposition 5.1], this can not happen. This finishes the proof of Lemma 2.1. $\square$

Lemma 2.2. *The group G embeds in $Af_+(\mathbb{R})$, the group of (orientation preserving) affine homeomorphism of the real line.*

Proof: We first embed G_1. Fix $a \in G_1$, $a \neq id$. Define $\varphi_a : G_1 \to Af_+(\mathbb{R})$ by declaring $\varphi_a(a)(x) = x+1$, and if $a' \in G_1$ is such that $(a')^q = a^p$, we declare $\varphi_a(a')(x) = x + p/q$. Showing that φ_a is an injective homomorphism is routine.

Now let $b \in G$ such that $\langle bG_1 \rangle = G/G_1$. Let $1 \neq r \in \mathbb{Q}$ such that $ba'b^{-1} = (a')^r$ for every $a' \in G_1$. Since G is bi-orderable we have that $r > 0$, and changing b by b^{-1} if necessary, we may assume that $r > 1$. Then, given $w \in G$, there is a unique $n \in \mathbb{Z}$ and a unique $\overline{w} \in G_1$ such that $w = b^n \overline{w}$.

Define $\varphi_{b,a} : G \to Af_+(\mathbb{R})$ by $\varphi_{b,a}(b^n\overline{w}) := H_r^{(n)} \circ \varphi_a(\overline{w})$, where $H_r(x) := rx$, and $H_r^{(n)}$ is the n-th composition of H_r (by convention $H_r^{(0)}(x) = x$). We claim that $\varphi_{b,a}$ is an injective homomorphism.

Indeed, let $w_1, w_2 \in G$, $w_1 = b^{n_1}\overline{w}_1$, $w_2 = b^{n_2}\overline{w}_2$. Let $r_1 \in \mathbb{Q}$ be such that $\varphi_a(\overline{w}_1)(x) = x+ r_1$. Then $H_r^{(n)} \circ \varphi_a(b^{-n}\overline{w}_1 b^n)(x) = H_r^{(n)} \circ \varphi_a(\overline{w}_1^{(1/r)^n})(x) = r^n(x + (1/r)^n r_1) = \varphi_a(\overline{w}_1) \circ H_r^{(n)}(x)$, for all $n \in \mathbb{Z}$. Thus

$$\begin{aligned}\varphi_{b,a}(w_1w_2) &= \varphi_{b,a}(b^{n_1}b^{n_2}\, b^{-n_2}\overline{w}_1 b^{n_2}\overline{w}_2) = H_r^{(n_1)} \circ H_r^{(n_2)} \circ \varphi_a(b^{-n_2}\overline{w}_1 b^{n_2}) \circ \varphi_a(\overline{w}_2) \\ &= H_r^{(n_1)} \circ \varphi_a(\overline{w}_1) \circ H_r^{(n_2)} \circ \varphi_a(\overline{w}_2) = \varphi_{b,a}(w_1) \circ \varphi_{b,a}(w_2).\end{aligned}$$

So $\varphi_{b,a}$ is a homomorphism. To see that it is injective, suppose that $\varphi_{b,a}(w_1)(x) = \varphi_{b,a}(b^{n_1}\overline{w}_1)(x) = r^n\, x + r^n r_1 = x$ for all $x \in \mathbb{R}$. Then $n = 0$ and $r_1 = 0$, showing that $w_1 = id$. This finishes the proof of Lemma 2.2. $\square$

Once the embedding $\varphi := \varphi_{b,a} : G \to Af_+(\mathbb{R})$ is fixed, we can associate to each irrational number ε an *induced left-ordering* $\preceq_\varepsilon$ on G whose set of positive elements is defined by $\{g \in G \mid \varphi(g)(\varepsilon) > \varepsilon\}$. When ε is rational, the preceding set defines only a partial ordering. However, in this case the stabilizer of the point ε is isomorphic to $\mathbb{Z}$, and hence this partial ordering may be completed to two total left-orderings $\preceq_\varepsilon^+$ and $\preceq_\varepsilon^-$. These orderings were introduced by Smirnov in [16]. Once the representation φ is fixed, we call these orderings, together with its corresponding reverse orderings, Smirnov-type orderings. (By definition the reverse ordering of $\preceq$, denoted $\overline{\preceq}$, satisfies $id \,\overline{\prec}\, g$ if and only if $id \prec g^{-1}$.)

Besides the Smirnov-type orderings on G, there are four Conradian (actually bi-invariant!) orderings. Since G_1 is always convex in a Conradian ordering, $b^n a^s \in G$, $n \neq 0$, is positive if and only if b is positive. Then it is not hard to check that the four Conradian orderings are the following:

1) $\preceq_{C_1}$, defined by $id \prec_{C_1} b^n a^s$ ($n \in \mathbb{Z}$, $s \in \mathbb{Q}$) if and only if $n \geq 1$, or $n = 0$ and $s > 0$.
2) $\preceq_{C_2}$, defined by $id \prec_{C_2} b^n a^s$ if and only if $n \leq -1$, or $n = 0$ and $s > 0$.
3) $\preceq_{C_3} = \overline{\preceq}_{C_1}$.
4) $\preceq_{C_4} = \overline{\preceq}_{C_2}$.

Proposition 2.3. *Let $U \subseteq \mathcal{LO}(G)$ be the set consisting of the four Conradian orderings together with the Smirnov-type orderings. Then any ordering in U is non-isolated in U.*

Proof: We first show that the Conradian orderings are non-isolated.

We claim that $\preceq_\varepsilon \to \preceq_{C_1}$ when $\varepsilon \to \infty$. For this it suffices to see that any positive element in the $\preceq_{C_1}$ ordering becomes $\preceq_\varepsilon$-positive for any ε is large enough.

By definition of $\preceq_\varepsilon$ we have that $id \prec_\varepsilon b^n a^s$ if and only if $r^n(\varepsilon + s) = \varphi(b^n a^s)(\varepsilon) > \varepsilon$, where $r > 1$. Now, assume that $id \prec_{C_1} b^n a^s$. If $n = 0$, then $s > 0$ and $\varepsilon + s > \varepsilon$. If $n \geq 1$, then $r^n(\varepsilon + s) > \varepsilon$ for $\varepsilon > \frac{-r^n s}{r^n - 1}$. So the claim follows.

For approximating the other three Conradian orderings, we first note that, arguing just as before, we have $\preceq_\varepsilon \to \preceq_{C_2}$ when $\varepsilon \to -\infty$. Finally, the other two Conradian orderings $\overline{\preceq}_{C_1}$ and $\overline{\preceq}_{C_2}$ are approximated by $\overline{\preceq}_\varepsilon$ when $\varepsilon \to \infty$ and $\varepsilon \to -\infty$ respectively.

Now let $\preceq_S$ be an Smirnov-type ordering and let $\{g_1, \ldots, g_n\}$ be a set of $\preceq_S$-positive elements.

Suppose first that $\preceq_S$ equals $\preceq_\varepsilon$, where ε has free orbit. Then we have that $\varphi(g_i)(\varepsilon) > \varepsilon$ for all $1 \leq i \leq n$. Thus, if ε' is such that $\varepsilon < \varepsilon' < \min\{\varphi(g_i)(\varepsilon)\}$, $1 \leq i \leq n$, then we still have that $\varphi(g_i)(\varepsilon') > \varepsilon'$, hence $g_i \succ_{\varepsilon'} id$ for $1 \leq i \leq n$. To see that $\preceq_{\varepsilon'} \neq \preceq_\varepsilon$, first notice that $\varphi(G_1)(x)$ is dense in $\mathbb{R}$ for all $x \in \mathbb{R}$. In particular, taking $g \in G_1$ such that $\varepsilon < \varphi(g)(0) < \varepsilon'$, we have that $\varphi(gb^ng^{-1})(\varepsilon) = \varphi(g)(r^n\varphi(g)^{-1}(\varepsilon)) = r^n\varphi(g)^{-1}(\varepsilon) + \varphi(g)(0)$. Since $\varphi(g)^{-1}(\varepsilon) < 0$ we have that for n large enough $gb^ng^{-1} \prec_\varepsilon id$. The same argument shows that $gb^ng^{-1} \succ_{\varepsilon'} id$. Therefore $\preceq_\varepsilon$ and $\preceq_{\varepsilon'}$ are distinct.

The remaining case is when $\preceq_S = \preceq_\varepsilon^{\pm}$. In this case we can order the set $\{g_1, \ldots, g_n\}$ such that there is i_0 with $\varphi(g_i)(\varepsilon) > \varepsilon$ for $1 \leq i \leq i_0$, and $\varphi(g_i)(\varepsilon) = \varepsilon$ for $i_0 + 1 \leq i \leq n$. That is $g_i \in Stab(\varepsilon) \simeq \mathbb{Z}$ for $i_0 + 1 \leq i \leq n$. Let $\varepsilon' > \varepsilon$.

We claim that either $\varphi(g_i)(\varepsilon') > \varepsilon'$ for all $i_0 + 1 \leq i \leq n$ or $\varphi(g_i)(\varepsilon') < \varepsilon'$ for all $i_0 + 1 \leq i \leq n$. Indeed, since φ gives an affine action, it can not be the case that a non-trivial element of G fixes two points. So we have that $\varphi(g_i)(\varepsilon') \neq \varepsilon'$ for each $i_0 + 1 \leq i \leq n$. Now, suppose for a contradiction that there are g_{i_0}, $g_{i_1} \in Stab(\varepsilon)$ with $g_{i_0}(\varepsilon') < \varepsilon'$ and $g_{i_1}(\varepsilon') > \varepsilon'$. Let $n, m \in \mathbb{N}$ be such that $g_{i_0}^n = g_{i_1}^m$. Then $\varepsilon' < \varphi(g_{i_1})^m(\varepsilon') = \varphi(g_{i_0})^n(\varepsilon') < \varepsilon'$. A contradiction. So the claim follows.

Now assume that $\varphi(g_i)(\varepsilon') > \varepsilon'$, for all $i_0 + 1 \leq i \leq n$. If, in addition $\varepsilon < \varepsilon' < \min\{\varphi(g_i)(\varepsilon)\}$ with $1 \leq i \leq i_0$, then $g_i \succ_{\varepsilon'} id$ for $1 \leq i \leq n$, showing that $\preceq_S$ is non-isolated. In the case where $\varphi(g_i)(\varepsilon') < \varepsilon'$ for all $i_0 + 1 \leq i \leq n$, we let $\tilde{\varepsilon}$ such that $\max\{\varphi(g_i)^{-1}(\varepsilon)\} < \tilde{\varepsilon} < \varepsilon$ for $1 \leq i \leq i_0$. Then we have that $g_i \succ_{\tilde{\varepsilon}} id$ for $1 \leq i \leq n$. This shows that, in any case, $\preceq_S = \preceq_\varepsilon^{\pm}$ is non-isolated in U. $\square$

The following theorem shows that the space of left-orderings of G is made up by the Smirnov-type orderings together with the Conradian orderings. This generalizes [14, Theorem 1.2].

Theorem 2.4. *Suppose G is a non Abelian group with rational series of length 2. If G is bi-orderable, then its space of left-orderings has no isolated points. Moreover, every non-Conradian ordering is equal to an induced, Smirnov-type, ordering arising from an affine action of G over $\mathbb{R}$ given by φ above.*

To prove Theorem 2.4, we will use the ideas (and notation) involved in the following well-known orderability criterion (see [5, Theorem 6.8], [11, §2.2.3], or [12, Proposition 2.1] for further details).

Proposition 2.5. *For a countable infinite group Γ, the following two properties are equivalent:*

– Γ *is left-orderable,*

– Γ *acts faithfully on the real line by orientation preserving homeomorphisms.*

Sketch of proof: The fact that a group of orientation preserving homeomorphisms of the real line is left-orderable is easy and may be found also in [8, Theorem 3.4.1]. In what follows, we will not make use of this.

For the converse, we construct what is called *the dynamical realization of a left-ordering.* Let $\preceq$ be a left-ordering on Γ. Fix an enumeration $(g_i)_{i\geq 0}$ of Γ, and let $t(g_0) = 0$. We shall define an order-preserving map $t : \Gamma \to \mathbb{R}$ by induction. Suppose that $t(g_0), t(g_1), \ldots, t(g_i)$ have been already defined. Then if g_{i+1} is greater (resp. smaller) than all $g_0, \ldots, g_i$, we define $t(g_{i+1}) = max\{t(g_0), \ldots, t(g_i)\} + 1$ (resp. $min\{t(g_0), \ldots, t(g_i)\} - 1$). If g_{i+1} is neither greater nor smaller than all $g_0, \ldots, g_i$, then there are $g_n, g_m \in \{g_0, \ldots, g_i\}$ such that $g_n \prec g_{i+1} \prec g_m$ and no g_j is between g_n, g_m for $0 \leq j \leq i$. Then we put $t(g_{i+1}) = (t(g_n) + t(g_m))/2$.

Note that Γ acts naturally on $t(\Gamma)$ by $g(t(g_i)) = t(gg_i)$. It is not difficult to see that this action extends continuously to the closure of $t(\Gamma)$. Finally, one can extend the action to the whole real line by declaring the map g to be affine on each interval in the complement of $t(\Gamma)$. □

Remark 2.6. As constructed above, the dynamical realization depends not only on the left-ordering $\preceq$, but also on the enumeration $(g_i)_{i\geq 0}$. Nevertheless, it is not hard to check that dynamical realizations associated to different enumerations (but the same ordering) are *topologically conjugate.*[3] Thus, up to topological conjugacy, the dynamical realization depends only on the ordering $\preceq$ of Γ.

An important property of dynamical realizations is that they do not admit global fixed points (*i.e.,* no point is stabilized by the whole group). Another important property is that $g \succ id$ if and only if $g(t(id)) > t(id)$, which allows us to recover the left-ordering from the dynamical realization.

Proof of Theorem 2.4: First fix $a \in G_1$ and $b \in G$ exactly as above, that is, such that $bab^{-1} = a^r$, where $r \in \mathbb{Q}$, $r > 1$, and $\varphi(a)(x) = x + 1$, $\varphi(b)(x) = rx$. Now let $\preceq$ be a left-ordering on G, and consider its dynamical realization. To prove Theorem 2.4, we will distinguish two cases:

Case 1. The element $a \in G$ is cofinal (that is, for every $g \in G$, there are n_1, $n_2 \in \mathbb{Z}$ such that $a^{n_1} \prec g \prec a^{n_2}$).

Note that in a Conradian ordering G_1 is convex. So a can not be cofinal. Thus, in this case we have to prove that $\preceq$ is an Smirnov-type ordering.

For the next two claims, recall that for any measure μ on a measurable space X and any measurable function $f : X \to X$, the *push-forward measure* $f_*(\mu)$ is defined by $f_*(\mu)(A) = \mu(f^{-1}(A))$, where $A \subseteq X$ is a measurable subset. Note that $f_*(\mu)$ is trivial if and only if μ is trivial. Moreover, one has $(fg)_*(\mu) = f_*(g_*(\mu))$ for all measurable functions f, g.

Similarly, the *push-backward measure* $f^*(\mu)$ is defined by $f^*(\mu)(A) = \mu(f(A))$.

[3]Two actions $\phi_1 : \Gamma \to \mathrm{Homeo}_+(\mathbb{R})$ and $\phi_2 : \Gamma \to \mathrm{Homeo}_+(\mathbb{R})$ are topologically conjugate if there exists $\varphi \in \mathrm{Homeo}_+(\mathbb{R})$ such that $\varphi \circ \phi_1(g) = \phi_2(g) \circ \varphi$ for all $g \in \Gamma$.

Claim 1. The subgroup G_1 preserves a Radon measure ν (*i.e.,* a measure which is finite on compact sets) on the real line which is unique up to scalar multiplication and has no atoms.

Since a is cofinal and G_1 is rank-one Abelian, its action on the real line is *free* (that is, no point is fixed by any non-trivial element of G_1). By Hölder's theorem (see [5, Theorem 6.10] or [11, §2.2]), the action of G_1 is semi-conjugated to a group of translations. More precisely, there exists a non-decreasing, continuous, surjective function $\rho\colon \mathbb{R} \to \mathbb{R}$ such that, to each $g \in G_1$, one may associate a translation parameter c_g so that, for all $x \in \mathbb{R}$,

$$\rho(g(x)) = \rho(x) + c_g.$$

Now since the Lebesgue measure *Leb* on the real line is invariant by translations, the *push-backward measure* $\nu = \rho^*(Leb)$ is invariant by G_1. Since *Leb* is a Radon measure without atoms, this is also the case for ν.

To see the uniqueness of ν up to scalar multiple we follow [11, §2.2.5]. Given any measure μ, invariant by the action (in this case) of G_1, we define the associated *translation number homomorphism* $\tau_\mu : G_1 \to \mathbb{R}$, by

$$\tau_\mu(g) = \begin{cases} \mu([x, g(x)]) & \text{if } g(x) > x, \\ 0 & \text{if } g(x) = x, \\ -\mu([g(x), x]) & \text{if } g(x) < x. \end{cases}$$

One easily checks that this definition is independent of $x \in \mathbb{R}$, and that the kernel of τ_μ coincides with the elements having fixed points, which in this case is just the identity of G_1. Now, by [11, Proposition 2.2.38], to prove the uniqueness of ν, it is enough to show that, for any non-trivial μ, $\tau_\mu(G_1)$ is dense in $\mathbb{R}$. But since G_1 is rank-one Abelian, and $G_1 \not\simeq \mathbb{Z}$, any non-trivial homomorphism from G_1 to $\mathbb{R}$ has a dense image. In particular $\tau_\mu(G_1)$ is dense in $\mathbb{R}$. So Claim 1 follows.

Claim 2. For some $\lambda \neq 1$, we have $b_*(\nu) = \lambda\nu$.

Since $G_1 \lhd G$, for any $a' \in G_1$ and all measurable $A \subset \mathbb{R}$ we must have

$$b_*(\nu)(a'(A)) = \nu(b^{-1}a'(A)) = \nu(\bar{a}(b^{-1}(A))) = \nu(b^{-1}(A)) = b_*(\nu)((A))$$

for some $\bar{a} \in G_1$. (Actually, $a' = \bar{a}^r$.) Thus $b_*(\nu)$ is a measure that is invariant by G_1. The uniqueness of the G_1-invariant measure up to scalar factor yields $b_*(\nu) = \lambda\nu$ for some $\lambda > 0$. Assume for a contradiction that λ equals 1. Then the whole group G preserves ν. In this case, there is a *translation number homomorphism* $\tau_\nu\colon G \to \mathbb{R}$ defined by

$$\tau_\nu(g) = \begin{cases} \nu([x, g(x)]) & \text{if } g(x) < x, \\ 0 & \text{if } g(x) = x, \\ -\nu([g(x), x]) & \text{if } g(x) < x. \end{cases}$$

The kernel of τ_ν must contain the commutator subgroup of G, and, since $a^{r-1} = [a, b] \in [G, G]$, we have that $\tau_\nu(a^{r-1}) = 0$, hence $\tau_\nu(a) = 0$. Nevertheless, this is impossible, since the kernel of τ_ν coincides with the set of elements having fixed points on the real line (see [11, §2.2.5]). So Claim 2 is proved.

By Claims 1 and 2, for each $g \in G$ we have $g_*(\nu) = \lambda_g(\nu)$ for some $\lambda_g > 0$. Moreover, $\lambda_a = 1$ and $\lambda_b = \lambda \neq 1$. Note that, as $(fg)_*(\nu) = f_*(g_*(\nu))$, the correspondence $g \to \lambda_g$ is a group homomorphism from G to $\mathbb{R}_+$, the group of positive real numbers under multiplication. Since G_1 is in the kernel of this homomorphism and any $g \in G$ is of the from $b^n a^s$ for $n \in \mathbb{Z},\ s \in \mathbb{Q}$, we have that the kernel of this homomorphism is exactly G_1.

Lemma 2.7. *Let $A : G \to Af_+(\mathbb{R})$, $g \to A_g$, be defined by*

$$A_g(x) = \frac{1}{\lambda_g} x + \frac{sgn(g)}{\lambda_g}\, \nu([t(g^{-1}), t(id)]),$$

where $sgn(g) = \pm 1$ is the sign of g in $\preceq$ (that is, $sgn(g) = 1$ if g is non-negative, and $sgn(g) = -1$ if g is negative.). Then A is an injective homomorphism.

Proof: For $g, h \in G$ both positive in $\preceq$, we compute

$$\begin{aligned}
A_{gh}(x) &= \frac{1}{\lambda_{gh}} x + \frac{1}{\lambda_{gh}} \nu([t((gh)^{-1}), t(id)]) \\
&= \frac{1}{\lambda_g \lambda_h} x + \frac{1}{\lambda_g \lambda_h} \left[(h_* \nu)([t(g^{-1}), t(h)]) \right] \\
&= \frac{1}{\lambda_g \lambda_h} x + \frac{1}{\lambda_g \lambda_h} \left[\lambda_h \nu([t(g^{-1}), t(id)]) + \nu([t(h^{-1}), t(id)]) \right] \\
&= \frac{1}{\lambda_g \lambda_h} x + \frac{1}{\lambda_g} \nu([t(g^{-1}), t(id)]) + \frac{1}{\lambda_g \lambda_h} \nu([t(h^{-1}), t(id)]) \\
&= A_g(A_h(x)).
\end{aligned}$$

The other cases can be treated analogously.

Now, assume that $A_g(x) = x$ for some non-trivial $g \in G$. Then $\lambda_g = 1$. In particular $g \in G_1$, since the kernel of the application $g \to \lambda_g$ is G_1. But in this case we have that g has no fixed point, so assuming that $0 = \lambda_g^{n-1} \nu([t(g^{-1}), t(id)] = \nu([t(g^{-n}), t(id)]$ implies ν is the trivial measure. This contradiction settles Lemma 2.7 . □

Now, for $x \in \mathbb{R}$, let $F(x) = sgn(x - t(id)) \cdot \nu([t(id), x])$. (Note that $F(t(id)) = 0$.) By semi-conjugating the dynamical realization by F we (re)obtain the faithful representation $A \colon G \to Af_+(\mathbb{R})$. More precisely, for all $g \in G$ and all $x \in \mathbb{R}$ we have

$$F(g(x)) = A_g(F(x)) \tag{2}$$

For instance, if $x > t(id)$ and $g \succ id$, then

$$\begin{aligned}
F(g(x)) &= \nu([t(id), g(x)]) \\
&= \frac{1}{\lambda_g} \nu([t(g^{-1}), x]) \\
&= \frac{1}{\lambda_g} \nu([t(g^{-1}), t(id)]) + \frac{1}{\lambda_g} \nu([t(id), x]) \\
&= \frac{1}{\lambda_g} F(x) + \frac{1}{\lambda_g} \nu([t(g^{-1}), t(id)]).
\end{aligned}$$

The action A induces a (perhaps partial) left-ordering $\preceq_A$, namely $g \succ_A id$ if and only if $A_g(0) > 0$. Note that equation (2) implies that for every $g \in G_1$, $g \succ id$, we have $A_g(0) > 0$ so $g \succ_A id$, and for every $f \in G$ such that $A_f(0) > 0$, we have $f \succ id$. In particular, if the orbit under A of 0 is free (that is, for every non-trivial element $g \in G$, we have $A_g(0) \neq 0$), then (2) yields that $\preceq_A$ is total and coincides with $\preceq$ (our original ordering).

If the orbit of 0 is not free (this may arise for example when $t(id)$ does not belong to the support of ν), then the stabilizer of 0 under the action of A is isomorphic to $\mathbb{Z}$. Therefore, $\preceq$ coincides with either $\preceq_A^+$ or $\preceq_A^-$ (the definition of $\preceq_A^\pm$ is similar to that of $\preceq_\varepsilon^\pm$ above).

At this point we have that $\preceq$ can be realized as an induced ordering from the action given by A. Therefore arguing as in the proof of Proposition 2.3 we have that $\preceq_A$, and so $\preceq$, is non-isolated.

To show that $\preceq$ is an Smirnov-type ordering, we need to determine all possible embeddings of G into the affine group. Recall that $bab^{-1} = a^r$, $r = p/q > 1$.

Lemma 2.8. *Every faithful representation of G in the affine group is given by*

$$a \sim \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \quad b \sim \begin{pmatrix} r & \beta \\ 0 & 1 \end{pmatrix}$$

for some $\alpha \neq 0$ and $\beta \in \mathbb{R}$.

Proof: Arguing as in Lemma 2.2 one may check that $\varphi'_{a,b} : \{a, b\} \to Af_+(\mathbb{R})$ defined by $\varphi'_{a,b}(a)(x) = x + \alpha$ and $\varphi'_{a,b}(b)(x) = rx + \beta$ may be (uniquely) extended to an homomorphic embedding $\varphi'_{a,b} : G \to Af_+(\mathbb{R})$. Conversely, let

$$a \sim \begin{pmatrix} s & \alpha \\ 0 & 1 \end{pmatrix}, \quad b \sim \begin{pmatrix} t & \beta \\ 0 & 1 \end{pmatrix}$$

be a representation. Since we are dealing with orientation preserving affine maps, s, t are positive real numbers. Moreover, the following equality must hold:

$$a^p \sim \begin{pmatrix} s^p & s^{p-1}\alpha + \ldots + s\alpha + \alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} s^q & s^{q-1}\alpha t + s^{q-2}\alpha t + \ldots + \alpha t - s^q\beta + \beta \\ 0 & 1 \end{pmatrix} \sim ba^qb^{-1}.$$

Thus $s = 1$, $t = p/q = r$. Finally, since the representation is faithful, $\alpha \neq 0$. □

Let α, β be such that $A_a(x) = x + \alpha$ and $A_b(x) = rx + \beta$. We claim that if the stabilizer of 0 under A is trivial –which implies in particular that $\beta \neq 0$– , then $\preceq_A$ (and hence $\preceq$) coincides with $\preceq_\varepsilon$ if $\alpha > 0$ (resp. $\overline{\preceq}_\varepsilon$ if $\alpha < 0$), where $\varepsilon = \frac{\beta}{(r-1)\alpha}$. Indeed, if $\alpha > 0$, then for each $g = b^n a^s \in G$, $s \in \mathbb{Q}$, we have $A_g(0) = r^n s\alpha + \beta\frac{r^n-1}{r-1}$. Hence $A_g(0) > 0$ holds if and only if

$$r^n \frac{\beta}{(r-1)\alpha} + r^n s > \frac{\beta}{(r-1)\alpha}.$$

Letting $\varepsilon := \frac{\beta}{(r-1)\alpha}$, one easily checks that the preceding inequality is equivalent to $g \succ_\varepsilon id$. The claim now follows.

In the case where the stabilizer of 0 under A is isomorphic to $\mathbb{Z}$, similar arguments to those given above show that $\preceq$ coincides with either $\preceq_\varepsilon^+$, or $\preceq_\varepsilon^-$, or $\overline{\preceq}_\varepsilon^+$, or $\overline{\preceq}_\varepsilon^-$, where ε again equals $\frac{\beta}{(r-1)\alpha}$.

Case 2. The element $a \in G$ is not cofinal.

In this case, for the dynamical realization of $\preceq$, the set of fixed points of a, denoted $Fix(a)$, is non-empty. We claim that $b(Fix(a)) = Fix(a)$. Indeed, let $r = p/q$, and let $x \in Fix(a)$. We have

$$a^p(b(x)) = a^p b(x) = ba^q(x) = b(x)\,.$$

Hence $a^p(b(x)) = b(x)$, which implies that $a(b(x)) = b(x)$ as asserted. Observe that since there is no global fixed point for the dynamical realization, we must have $b(x) \neq x$, for all $x \in Fix(a)$. Note also that, since G_1 is rank-one Abelian group, $Fix(a) = Fix(G_1)$.

Now let $x_{-1} = \inf\{t(g) \mid g \in G_1\}$ and $x_1 = \sup\{t(g) \mid g \in G_1\}$. It is easy to see that x_{-1} and x_1 are fixed points of G_1. Moreover, x_{-1} (resp. x_1) is the first fixed point of a on the left (resp. right) of $t(id)$. In particular, $b((x_{-1}, x_1)) \cap (x_{-1}, x_1) = \emptyset$, since otherwise one may create a fixed point inside (x_{-1}, x_1). Taking the *reverse* ordering if necessary, we may assume $b \succ id$. In particular, we have that $b(x_{-1}) \geq x_1$.

We now claim that G_1 is a convex subgroup. First note that, by the definition of the dynamical realization, for every $g \in G$ we have $t(g) = g(t(id))$. Then, it follows that for every $g \in G_1$, $t(g) \in (x_{-1}, x_1)$. Now let $m, s \in \mathbb{Z}$ and $g \in G_1$ be such that $id \prec b^m g \prec a^s$. Then we have $t(id) < b^m(t(g)) < t(a^s) < x_1$. Since $b(x_{-1}) \geq x_1$, this easily yields $m = 0$, that is, $b^m g = g \in G_1$.

We have thus proved that G_1 is a convex (normal) subgroup of G. Since the quotient G/G_1 is isomorphic to $\mathbb{Z}$, an almost direct application of Theorem 1.1 shows that the ordering $\preceq$ is Conradian. This concludes the proof of Theorem 2.4. $\square$

Remark 2.9. It follows from Theorem 2.4 and Proposition 2.3 that no left-ordering is isolated in $\mathcal{LO}(G)$. Therefore, since any group with normal rational series is countable, $\mathcal{LO}(G)$ is a totally disconnected Hausdorff and compact metric space, thus homeomorphic to the Cantor set.

Remark 2.10. The above method of proof also gives a complete classification –up to topological semiconjugacy– of all actions of G by orientation-preserving homeomorphisms of the real line (compare [13]). In particular, all these actions come from left-orderings on the group (compare with Question 2.4 in [12] and the comments before it).

3 The general case

3.1 A technical proposition

The main objective of this section is to prove the following

Proposition 3.1. *Let G be a group with only finitely many C-orderings, and let H be its maximal convex subgroup (with respect to any C-ordering). Then H is a Tararin group, that is, a group with only finitely many left-orderings.*

Note that the existence of a maximal convex subgroup follows from Theorem 1.2. Note also that Proposition 3.1 implies that no group with only finitely many C-orderings, whose rational series has length at least 3, is bi-orderable (see also [14, Proposition 3.2]).

The proof of Proposition 3.1 is a direct consequence of the following

Lemma 3.2. *Let G be a group with only finitely many C-orderings whose rational series has length at least three:*

$$\{id\} = G_0 \lhd G_1 \lhd G_2 \lhd \ldots \lhd G_n = G\,, \quad n \geq 3. \tag{3}$$

Then given $a \in G_1$ and $b \in G_i$, $i \leq n-1$, we have that $bab^{-1} = a^{\varepsilon}$, $\varepsilon = \pm 1$.

Proof: We shall proceed by induction on i. For $i = 0, 1$ the conclusion is obvious. We work the case $i = 2$. Let $b \in G_2$, and suppose that $bab^{-1} = a^r$, where $r \neq \pm 1$ is rational. Clearly this implies that $b^n ab^{-n} = a^{r^n}$ for all $n \in \mathbb{Z}$.

Since G_3/G_1 is non-Abelian, there exists $c \in G_3$ such that $cb^pc^{-1} = b^q w$, with $p \neq q$ integers and $w \in G_1$. Note that $wa = aw$. We let $t \in \mathbb{Q}$ be such that $cac^{-1} = a^t$. Then we have

$$a^{r^q} = b^q ab^{-q} = b^q\, waw^{-1}b^{-q} = cb^pc^{-1}a\, cb^{-p}c^{-1} = cb^pa^{1/t}b^{-p}c^{-1} = ca^{\frac{r^p}{t}}c^{-1} = a^{r^p},$$

which is impossible since $r \neq \pm 1$ and $p \neq q$. Thus the case $i = 2$ is settled.

Now assume, as induction hypothesis, that for any $w \in G_{i-1}$ we have that $waw^{-1} = a^{\varepsilon}$, $\varepsilon = \pm 1$. Suppose also that there exists $b \in G_i$ such that $bab^{-1} = a^r$, $r \neq \pm 1$. As before, we have that $b^n ab^{-n} = a^{r^n}$ for all $n \in \mathbb{Z}$.

Let $c \in G_{i+1}$ such that $cb^pc^{-1} = b^q w$, with $p \neq q$ integers and $w \in G_{i-1}$. Let $t \in \mathbb{Q}$ be such that $cac^{-1} = a^t$. Then we have

$$a^{r^q} = b^q ab^{-q} = b^q\, w\, w^{-1}aw\, w^{-1}b^{-q} = cb^pc^{-1}a^{\varepsilon}\, cb^{-p}c^{-1} = cb^pa^{\varepsilon/t}b^{-p}c^{-1} = ca^{\frac{\varepsilon r^p}{t}}c^{-1} = a^{\varepsilon r^p},$$

which is impossible since $r \neq \pm 1$ and $p \neq q$ implies $|r^p| \neq |r^q|$. This finishes the proof of Lemma 3.2. $\square$

Proof of Proposition 3.1: Since in any Conradian ordering of G, the convex series is precisely the rational series, we have that $H = G_{n-1}$ in (3). So H has a rational normal series. Therefore, to prove that H is a Tararin group, we only need to check that no quotient G_i/G_{i-2}, $2 \leq i \leq n-1$, is bi-orderable.

Now, if in (3) we take the quotient by the normal and convex subgroup G_{i-2}, Lemma 3.2 implies that certain element in G_{i-1}/G_{i-2} is sent into its inverse by the action of some element in G_i/G_{i-2}. Thus G_i/G_{i-2} is non-bi-orderable. $\square$

Corollary 3.3. *A group G having only finitely many C-orderings, with rational series*

$$\{id\} \lhd G_1 \lhd \ldots \lhd G_{n-1} \lhd G_n = G,$$

is a Tararin group if and only if G/G_{n-2} is a Tararin group.

3.2 Proof of the Main Theorem

Let G be a group with rational series

$$\{id\} = G_0 \lhd G_1 \lhd \ldots \lhd G_{n-1} \lhd G_n = G, \quad n \geq 3,$$

such that no quotient G_i/G_{i-2} is Abelian. Moreover, assume G is not a Tararin group. Let $\preceq$ be a left-ordering on G. To show that $\preceq$ is non-isolated we will proceed by induction. Therefore, we assume as induction hypothesis that no group with only finitely many C-orderings, but infinitely many left-orderings, whose rational series has length less than n, has isolated left-orderings.

The main idea of the proof is to find a convex subgroup H such that either H has no isolated left-orderings or such that H is normal and G/H has no isolated left-orderings. Indeed, by Corollary 1.5, this is enough to show that $\preceq$ is non-isolated. We will see that the appropriate convex subgroup to look at is the *convex closure of* G_1 (with respect to $\preceq$), that is, the smallest convex subgroup that contains G_1.

For $x, y \in G$, consider the relation in G given by $x \sim y$ if and only if there are $g_1, g_2 \in G_1$ such that $g_1 x \preceq y \preceq g_2 x$. We check that $\sim$ is an equivalence relation. Clearly $x \sim x$ for all $x \in G$. If $x \sim y$ and $y \sim z$ then there are $g_1, g_2, g_1', g_2' \in G_1$ such that $g_1 x \preceq y \preceq g_2 x$ and $g_1' y \preceq z \preceq g_2' y$. Then $g_1' g_1 x \preceq z \preceq g_2' g_2 x$, so $x \sim z$. Finally $g_1 x \preceq y \preceq g_2 x$ implies $g_2^{-1} y \preceq x \preceq g_1^{-1} y$, so $x \sim y$ implies $y \sim x$.

Now let g, x, y in G be such that $x \sim y$, hence $g_1 x \preceq y \preceq g_2 x$, for some $g_1, g_2 \in G_1$. Then $g g_1 x \preceq g y \preceq g g_2 x$. Since G_1 is normal we have that $g g_1 x = g_1' g x$ and $g g_2 x = g_2' g x$, for some $g_1', g_2' \in G_1$. Therefore, $g_1' g x \preceq g y \preceq g_2' g x$, so $gx \sim gy$. That is, G preserves the equivalence relation $\sim$. Let $H := \{x \in G \mid x \sim id\}$.

<u>Claim 1:</u> For every $g \in G$ we have

$$gH \cap H = \begin{cases} \emptyset & \text{if } g \notin H, \\ H & \text{if } g \in H. \end{cases}$$

Indeed, if $g \in H$, then $g \in (gH \cap H)$. Now, since $x \sim y$ if and only if $gx \sim gy$, we have that $gH = H$. Now suppose g is such that there is $z \in gH \cap H$. Then $id \sim z \sim g$, which implies $g \in H$. So Claim 1 follows.

Claim 1 implies that H is a convex subgroup of G that contains G_1. Moreover, we have

<u>Claim 2:</u> The subgroup H is the convex closure of the subgroup G_1.

Indeed, let C denote the convex closure of G_1 in $\preceq$. Then H is a convex subgroup that contains G_1. Thus $C \subseteq H$.

To show that $H \subseteq C$ we just note that, by definition, for every $h \in H$, there are $g_1, g_2 \in G_1$ such that $g_1 \preceq h \preceq g_2$. So $H \subseteq C$, and Claim 2 follows.

Proceeding as in Lemma 2.1 we conclude that there exists $c \in G$ such that $c G_{n-1}$ generates the quotient G/G_{n-1}. We have

<u>Claim 3:</u> H/G_1 is either trivial or isomorphic to $\mathbb{Z}$.

By proposition 3.1, G_{n-1} is a Tararin group. Therefore, in the restriction of $\preceq$ to G_{n-1}, G_1 is convex. So we have that $H \cap G_{n-1} = G_1$. This means that for every $g \in G_{n-1} \setminus G_1$, $gH \cap H = \emptyset$.

Now, assume H/G_1 is non-trivial and let $g \in H \setminus G_1$. By the preceding paragraph we have that $g \notin G_{n-1}$. Therefore, $g = c^{m_1} w_{m_1}$, for $m_1 \in \mathbb{Z}$, $m_1 \neq 0$ and $w_{m_1} \in G_{n-1}$.

Let m_0 be the least positive $m \in \mathbb{Z}$ such that $c^m w_m \in H$, for $w_m \in G_{n-1}$. Then, by the minimality of m_0, we have that m_1 is a multiple of m_0, say $km_0 = m_1$. Letting $(c^{m_0} w_{m_0})^k = c^{m_0 k} \overline{w_{m_0}}$, we have that $(c^{m_0} w_{m_0})^{-k} c^m w_m = \overline{w_{m_0}}^{-1} w_m \in H$. Since $\overline{w_{m_0}}^{-1} w_m \in G_{n-1}$, we have that $\overline{w_{m_0}}^{-1} w_m \in G_1$. So we conclude that $(c_0^m w_{m_0})^k \, G_1 = c^m w_m \, G_1$, which proves our Claim 3.

We are now in position to finish the proof of the Main Theorem. According to Claim 3 above, we need to consider two cases.

Case 1: $H = G_1$.

In this case, G_1 is a convex normal subgroup of $\preceq$ and, since by induction hypothesis G/G_1 has no isolated left-orderings, $\preceq$ is non-isolated.

Case 2: $H/G_1 \simeq \mathbb{Z}$.

In this case, H has a rational series of length 2:

$$\{id\} = G_0 \lhd G_1 \lhd H.$$

We let $a \in G_1$, $a \neq id$, and $h \in H$ be such that hG_1 generates H/G_1. Let $r \in \mathbb{Q}$ be such that $hah^{-1} = a^r$. We have three subcases:

Subcase i) $r < 0$.

Clearly, in this subcase, H is non-bi-orderable. So H is a Tararin group and G_1 is convex in H. But, as proved in Claim 2, H is the convex closure of G_1. Therefore, this subcase does not arise.

Subcase ii) $r > 0$.

Since $r > 0$, we have that H is not a Tararin group, thus H has no isolated left-orderings. Therefore $\preceq$ is non-isolated.

Subcase iii) $r = 0$.

In this case, H is a rank-two Abelian group, so it has no isolated orderings. Hence $\preceq$ is non-isolated. This finishes the proof of the Main Theorem.

3.3 An illustrative example

This subsection is aimed to illustrate the different kinds of left-orderings that may appear in a group as above. To do this, we will consider a family of groups with eight C-orderings. We let $G(n) = \langle a, b, c \mid bab^{-1} = a^{-1}, cbc^{-1} = b^3, cac^{-1} = a^n \rangle$, where $n \in \mathbb{Z}$. It is easy to see that $G(n)$ has a rational series of length three,

$$\{id\} \lhd G_1 = \langle a \rangle \lhd G_2 = \langle a, b \rangle \lhd G(n).$$

In particular, in a Conradian ordering, G_1 is convex and normal.

Now we note that $G(n)/G_1 \simeq B(1,3)$, where $B(1,3) = \langle \beta, \gamma \mid \gamma\beta\gamma^{-1} = \beta^3 \rangle$ is a Baumslag-Solitar group, and the isomorphism is given by $c \to \gamma$, $b \to \beta$, $a \to id$. Now consider the (faithful) representation $\varphi : B(1,3) \to Homeo_+(\mathbb{R})$ of $B(1,3) \simeq G(n)/G_1$

into $Homeo_+(\mathbb{R})$ given by $\varphi(\beta)(x) = x+1$ and $\varphi(\gamma)(x) = 3x$. It is easy to see that, if $x \in \mathbb{R}$, then $Stab_{\varphi(B(1,3))}(x)$ is either trivial or isomorphic to $\mathbb{Z}$.

In particular, $Stab_{\varphi(B(1,3))}(\frac{-3k}{2}) = \langle \gamma\beta^k \rangle$, where $k \in \mathbb{Z}$. Thus $\langle \gamma\beta^k \rangle$ is convex in the induced ordering from $\frac{-3k}{2}$ (in the representation given by φ). Now, using the isomorphism $G(n)/G_1 \simeq B(1,3)$, we have induced an ordering on $G(n)/G_1$ with the property that $\langle cb^k\, G_1 \rangle$ is convex. We denote this left-ordering by $\preceq_2$. Now, extending $\preceq_2$ by the initial Conradian ordering on G_1, we have created an ordering $\preceq$ on $G(n)$ with the property that $H(n) = \langle a, cb^k \rangle$ is convex. Moreover, we have:

- If $n = 1$ and $k = 0$, then $H(n) = \langle a, c \rangle \leq G(n)$ is convex in $\preceq$ and $ca = ac$, as in Subcase *iii*) above.

- If $n \geq 2$, and $k = 0$, then $H(n) = \langle a, c \rangle \leq G(n)$ is convex in $\preceq$ and $cac^{-1} = a^2$, as in Subcase *ii*) above.

- If $n \leq -1$ and k is odd, then $H(n) = \langle a, cb^k \rangle \leq G(n)$ is convex and $cb^k\, a\, b^{-k} c^{-1} = a^{-n}$ (again) as in Subcase *ii*) above.

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